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# The travelling salesman problem on a randomly diluted lattice

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**Abstract.** We study the problem of a travelling salesman who must visit a randomly chosen subset of sites of a $d$-dimensional lattice. The average length of the shortest path per chosen site is $\alpha(q)$ where $(1-q)$ is the density of chosen sites. For a triangular lattice, we show that $\alpha(q)$ differs from 1 only by terms of order $q^5$. For the square lattice, we show that, to first order in $q$, optimal paths can be found from the dynamics of a model of a one-dimensional gas of kinks and antikinks. We find $\alpha(q) \leqslant 1 +$ terms of order $q^{3/2}$. We also obtain a constructive upper bound valid for all $q$, which gives $\alpha(q) \leqslant \sqrt{\frac{4}{3}}(1-q)^{-1/2}$ as $q$ tends to 1.

## 1. Introduction

The travelling salesman problem (TSP) is a well known optimisation problem. The object is to find the shortest route of a travelling salesman (TS) who must visit each of $N$ specified cities at least once, given the intercity distances. The problem belongs to the non-deterministic polynomial-time complete (NP-complete) class of problems: all known deterministic algorithms for finding the optimal route require a computational effort that increases exponentially with $N$ (Garey and Johnson 1979). Several heuristic algorithms for generating suboptimal tours are known and, if the number of cities is not too large, branch and bound methods can be used to find the optimal tours as well. For a comparative study of performance of several heuristics and a good review of other theoretical results, see Lawler *et al* (1985).

The definition of a finite-temperature problem (Kirkpatrick 1984) which has the shortest route as its ground state has led to a recent upsurge of interest in this problem. The method of simulated annealing (Kirkpatrick 1984) is an outcome of this idea, as is the realisation that the problem has many features in common with spin glasses (Kirkpatrick 1981, Vannimenus and Mézard 1984, Kirkpatrick and Toulouse 1985, Baskaran *et al* 1986, Fu and Anderson 1986, Mézard and Parisi 1986a, b, Sourlas 1986). Bonomi and Lutton (1984) have used simulated annealing to obtain a near-optimal solution for a configuration of 10 000 cities placed randomly on a plane. The lattice version of the TSP has been studied recently on a diluted lattice by Chakrabarti (1986) and on the Sierpinski gasket by Bradley (1986).

In this paper we shall consider the case where the cities to be visited are placed randomly on the vertices of a $d$-dimensional periodic lattice, the fractional number of

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sites chosen being $p$. The fractional number of sites that do not have to be visited is $q=1-p$. In one step, the salesman can move from a site of the lattice to any of the neighbouring sites.

Let the number of sites in the lattice be $N$. Let $\mathbb{P}(\mathbb{C})$ be the shortest tour for a given configuration $\mathbb{C}$ of chosen sites. The length of the path $l(\mathbb{P})$ is defined as the total number of steps in the path. $l(\mathbb{P})$ is a random variable depending on the configuration $\mathbb{C}$. For large $N$, $l(\mathbb{P})$ varies linearly with $N$. In fact, it can be shown (Beardwood *et al* 1959) that the limit

$$\alpha(q)=\lim_{N\to\infty}\frac{l(\mathbb{P})}{(1-q)N} \tag{1}$$

exists with probability 1. This represents the average length of the minimising path (measured in terms of lattice spacings) between two consecutive occupied sites in $\mathbb{P}$. $\alpha(q)$ depends on the lattice structure. In this paper, we obtain constructive upper bounds for $\alpha(q)$ and determine its qualitative behaviour near $q=0$ and $q=1$.

The plan of the paper is as follows. After a brief discussion of the general qualitative behaviour of $\alpha(q)$ for arbitrary lattices in § 2, we discuss the case of the triangular lattice in § 3. We show that in this case for $q$ near 0, $\alpha(q)$ differs from one only by terms of order $q^5$. For the square lattice, the problem is more difficult. In § 4, we describe two strategies which provide upper bounds to $\alpha(q)$ for all $q$, $0<q\leqslant 1$. In § 5 we give an explicit construction of a route on the square lattice which avoids vacancies to order $q$ and which can be viewed as defining kink-antikink dynamics with pair production at vacancies and pairwise annihilation and scattering.

## 2. Preliminaries

For any walk, the number of steps cannot be less than the number of sites visited (excluding the origin of the walk). Clearly, for all $q$, and all lattices

$$\alpha(q)\geqslant 1. \tag{2}$$

Also $\alpha(0)=1$ for hypercubic lattices. The optimal (non-unique) path in the $q=0$ case is a self-avoiding walk going through all the sites of the lattice. The number of such walks (called Hamilton walks) is known to grow exponentially with the size of the lattice (Kasteleyn 1963). The number of optimal paths presumably remains exponential in $N$, even for $q\neq 0$. The dependence of this number on $q$ is difficult to study and will not be discussed here.

The optimal path length cannot increase if a site is removed from the set of chosen sites. Since this quantity per lattice site is $(1-q)\alpha(q)$, we have

$$(\mathrm{d}/\mathrm{d}q)[(1-q)\alpha(q)]\leqslant 0. \tag{3}$$

This implies in particular that

$$(1-q)\alpha(q)\leqslant\alpha(0)=1. \tag{4}$$

This provides an upper bound $\alpha(q)\leqslant 1/(1-q)$ which is saturated in the trivial case $d=1$.

For $q\approx 1$, the average separation between occupied sites increases as $(1-q)^{-1/d}$ where $d$ is the dimension of the lattice. Since $\alpha(q)$ should scale linearly with the average separation then

$$\alpha(q)\simeq A(1-q)^{-1/d} \qquad \text{for } q\to 1 \tag{5}$$

where $A$ is a constant of proportionality. This has been proved rigorously by Beardwood *et al* (1959). When $p$ is non-zero, $\alpha(q)p^{1/d}$ is not constant due to the influence of the lattice structure. We expect (but have no proof) that $A$ is less than one and $\alpha(q)(1-q)^{1/d}$ is a monotonically decreasing function of $q$. For the continuum model with a Euclidean metric in two dimensions, Armour and Wheeler (1983) have shown that $A_E \leqslant 0.921$, where the subscript E refers to the Euclidean metric. $A_E$ is not known exactly. Numerical estimates from finite samples give (Beardwood *et al* 1959, Bonomi and Lutton 1984, Randelman and Grest 1986)

$$A_E \simeq 0.75. \tag{6}$$

When the distance is measured along bonds of the lattice (e.g. in equation (5)), the distance is larger than the Euclidean distance, and hence $A$ is larger than $A_E$. An upper bound for $A$ in (5) can be obtained in terms of $A_E$ as follows. For a given configuration $\mathbb{C}$ of sites, construct the Euclidean TS path. Then construct a TS path $\mathbb{P}_E$ on the lattice, visiting the cities in the same sequence. In a direction $\theta$ from the $x$ axis the lattice distance is $(|\cos\theta|+|\sin\theta|)$ times the Euclidean distance. Assuming the rotational symmetry of the Euclidean TS path, all directions of $\theta$ of the straight line joining two consecutive cities are equally likely. The average value of $(|\cos\theta|+|\sin\theta|)$ is $4/\pi$. This gives

$$A_E \leqslant A \leqslant 4A_E/\pi \simeq 0.96. \tag{7}$$

For $p \neq 1$, the inequality in (2) is strict as then, on any lattice, there exists a finite density of occupied sites having only one occupied neighbour. A salesman visiting these sites must either visit the occupied neighbour twice or visit at least once one of the unoccupied neighbours. For each such branch site, the contribution to excess path length (i.e. path length minus number of visited occupied sites) increases by at least one. On a lattice with coordination number $z$, the density of such sites is $zq^{z-1}p^2$. In addition, there are isolated occupied sites, with no occupied neighbours (density $pq^z$), which necessitate at least two visits to unoccupied neighbours. This gives

$$\alpha(q) \geqslant 1 + zp^2q^{z-1} + 2pq^z. \tag{8}$$

These are relatively large orders in $q$. We conjecture that this estimate correctly describes the behaviour near $q=0$. In the Taylor expansion of $\alpha(q)$ in powers of $q$

$$\alpha(q) = 1 + \sum_{i=1}^{\infty} C_i q^i \tag{9}$$

we must have

$$C_i = 0 \qquad \text{for} \qquad 1 \leqslant i \leqslant z-2. \tag{10}$$

For some lattices this conjecture can be proved by a constructive argument. The case of the triangular lattice is described in the next section.

Since on a hypercubic lattice $z$ is $2d$, (9) and (10) would imply that, as $d \to \infty$ for $q$ fixed, $\alpha(q)$ should approach one. In this connection it should be noted that the TSP on the Cayley tree (which represents the $d \to \infty$ limit in many other cases) is quite different. This is because the Cayley tree has no closed loops, so that a particular unoccupied site can be skipped only if it, and all its descendants, are unoccupied; the probability of this is small. An explicit formula for $\alpha(q)$ on a Cayley tree with $m$ generations (and coordination number $z$) can be worked out explicitly:

$$\alpha(q) = \frac{2(z-2)\sum_{k=0}^{m}(z-1)^k(1-Q_k)}{(z-1)^{m+1}-1} \tag{11}$$

where

$$\ln Q_k = \left(\frac{(z-1)^{m-k+1}-1}{z-2}\right) \ln q. \qquad (12)$$

Note that $\alpha(q=0)$ is not one as no Hamilton walks are possible on the Cayley tree.

## 3. The triangular lattice for $q$ near zero

Consider first the case $q=0$. We consider a special Hamiltonian walk on a large lattice (figure 1($a$)). The several strands shown in the figure are joined at the boundary of the lattice to form a single path, to be called the standard route. We shall ignore the surface contributions, as they are negligible in the thermodynamic limit.

If there is a single unoccupied site (the crossed site in the upper-right corner of figure 1($b$)), the standard route can be modified so that the salesman skips the unoccupied site, going directly from the preceding to the next site in a single step. If there are several unoccupied sites well separated from each other, the standard route can be locally modified to skip them independently. Clearly one path length is saved for each site omitted.

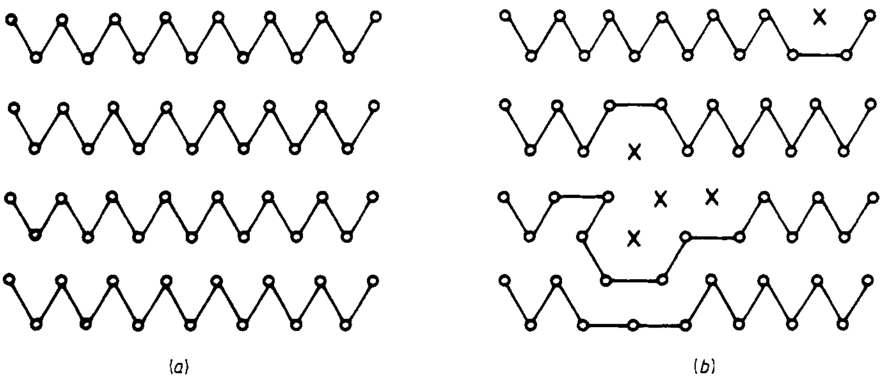


**Figure 1.** ($a$) The standard route on the triangular lattice. ($b$) An example of how small clusters of vacant sites can be avoided by locally deforming the standard route. Here the vacant sites are represented by crosses.

A similar local modification of the standard route can be found to accommodate small clusters of unoccupied sites. In figure 1($b$), we show how the standard route may be modified locally to skip a cluster of four unoccupied sites, saving four steps of path length.

There is only a finite number of configurations with up to four nearby unoccupied sites ('nearby' here does not necessarily imply neighbouring; only sufficiently close that the standard route cannot be modified independently to skip them). We found that it is possible to modify the standard route locally for each such configuration, skipping all four sites without having to visit any other site twice (details omitted).

To order $q^4$, only such configurations occur, and therefore $\alpha(q)$ is one on the triangular lattice to this order. This proves that (8) holds for the triangular lattice, for which $z=6$.

For the square lattice too (or other bipartite hypercubic lattices) we expect (9) and (10) to hold. However, we have to use non-local modifications of paths to prove this assertion. Because of strong odd-even effects on such lattices, a salesman can only be on odd sites after an odd number of steps. Hence a *local* modification of a path avoiding a single unoccupied site and decreasing the path length by one is clearly impossible. This makes the construction of paths avoiding well separated defects quite difficult. The strategy discussed in § 5 gives

$$\alpha(q) = 1 + \text{terms of order } q^{3/2}. \qquad (13)$$

This implies, in particular, that for the square lattice

$$C_1 = 0. \qquad (14)$$

## 4. Foliation on the square lattice

We now construct an upper bound for $\alpha(q)$ for the square lattice, using a variant of the foliation strategy discussed by Armour and Wheeler (1983) in the continuum case. Other lattices may be treated similarly but will not be discussed explicitly here.

The principal idea is to divide the lattice into strips of $L$ columns each and let the salesman move down each strip row by row, avoiding vacant sites when possible. The value of $L$ is chosen to minimise the average length of tour and depends on $q$.

We will discuss two foliation strategies denoted by F1 and F2 respectively. Figure 2 illustrates both, with walks on strips of width $L = 5$. In either case, the path consists of several horizontal steps (left or right) followed by a single vertical step down to the next row. Let $l_n$ and $r_n$ be the $x$ coordinate of the leftmost and rightmost occupied sites in the $n$th row. If the entire row is unoccupied, we define $l_n = L+1$ and $r_n = 0$. Let $x_n$ be the horizontal coordinate of the vertical step leaving the $n$th row.

In strategy F1, we define

$$x_n = \min(l_n, l_{n+1}, x_{n-1}) \qquad \text{if } n \text{ is odd} \qquad (15a)$$

and

$$x_n = \max(x_{n-1}, r_n, r_{n+1}) \qquad \text{if } n \text{ is even}. \qquad (15b)$$

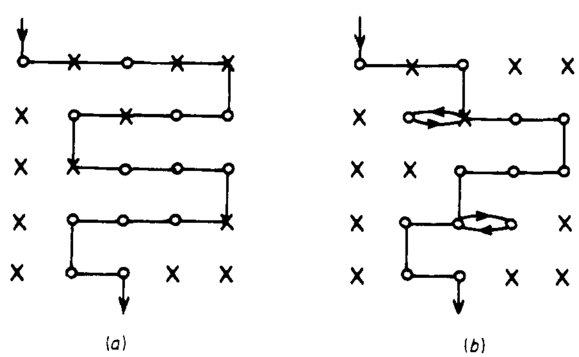


**Figure 2.** Foliation strategies F1 and F2 are illustrated. In the example shown, for the same configuration of unoccupied sites, the path in ($a$) using the strategy F1 is longer by two steps than that in ($b$) using strategy F2.

The salesman's path consists of leftward steps on odd rows (up to $x_{2n+1}$ on the $(2n+1)$th row) and rightward steps on even rows.

In strategy F2, no distinction is made between even and odd rows. On entering the $n$th row at $x_{n-1}$, the salesman first goes to the left or the right endpoint (of occupied sites) $l_n$ or $r_n$, whichever is closer. He then goes to the other endpoint (retracing some steps if necessary) and then takes a step down to the next row. Thus

$$x_n = r_n \qquad \text{if } \tfrac{1}{2}(r_n + l_n) < x_{n-1} \tag{16a}$$

$$= l_n \qquad \text{if } \tfrac{1}{2}(r_n + l_n) > x_{n-1}. \tag{16b}$$

If $x_n$ is exactly halfway between $l_n$ and $r_n$, then the exit point is chosen to be either of them with equal probability. If the entire row is unoccupied, then $x_n$ is equal to $x_{n-1}$.

In both strategies F1 and F2, it is clear that every occupied site in a strip is visited at least once. Of course, some unoccupied sites are also visited. Since $x_{n+1}$ depends only on $x_n$ and the configuration of row $n+1$ (and also row $n+2$ in the case of F1) the sequence of $x_n$ forms a Markov chain. Let the transition probability matrices for the strategies F1 and F2 be denoted by $T_1$ and $T_2$ respectively.

First consider $T_1$. We define a variable

$$y_n = x_n \qquad \text{when } n \text{ is odd} \tag{17a}$$

$$= L+1-x_n \qquad \text{when } n \text{ is even.} \tag{17b}$$

Then we have

$$T_1(y_{n+1}|y_n) = q^{2(y_{n+1}-1)}(1-q^2) \qquad \text{if } y_{n+1} < L+1-y_n \tag{18a}$$

$$= q^{2(y_{n+1}-1)} \qquad \text{if } y_{n+1} = L+1-y_n \tag{18b}$$

$$= 0 \qquad \text{otherwise.} \tag{18c}$$

By inspection, the steady state probabilities for $y_i$ are

$$\text{prob}(y_i = r) = Aq^{2r} \tag{19}$$

where $A$ is a normalisation constant. The path length saved is $2y_i$, in going from row $i$ to row $i+1$. The mean saving per row is†

$$S(L) = 2\left(\frac{q^2}{1-q^2} - \frac{Lq^{2L}}{1-q^{2L}}\right). \tag{20}$$

From $S(L)$ one can easily find the average path length per visited site for F1

$$\alpha_1(q) = (1 - S(L)/L)/p. \tag{21}$$

We choose $L$ so as to minimise $\alpha(q)$. We find that $L=2$ for $0<q<0.551$; $L=3$ for $0.763>q>0.551$ and $L=4$ for $0.855>q>0.763$. As $q\to 1$, the optimal value of $L$ is $\sqrt{3/p}$, and $\sqrt{p}\alpha_1(q)$ approaches $\sqrt{\frac{4}{3}}$ It is interesting to compare this value with that of Armour and Wheeler. Their optimal value of $L$ is the same as here, but $\sqrt{p}\alpha(q)$ is lower, 0.921, due to the different metric (Euclidean) used. In figure 3 we show $\sqrt{p}\alpha_1(q)$ against $q$. The strategy works best for $q$ close to 1 and performs relatively poorly for intermediate and low $q$.

† In the limit $L\to\infty$ only the first term of (20) survives, describing the path length saved by a 'directed salesman' (Chakrabarti 1986).

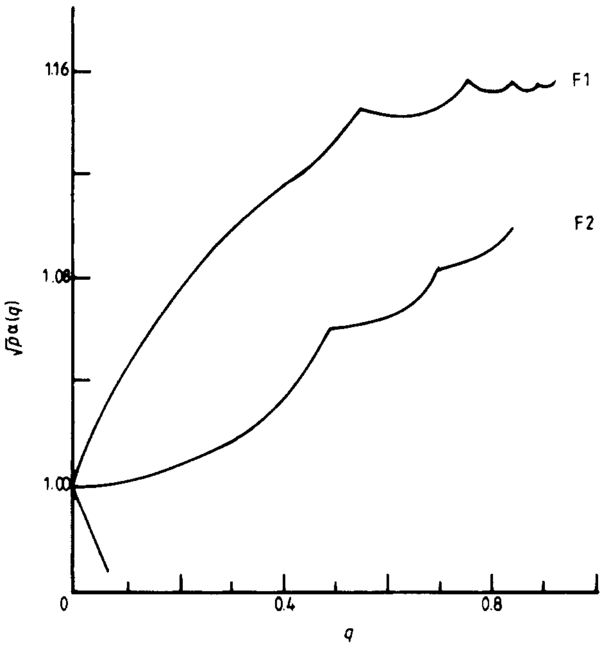


**Figure 3.** A comparison of the efficiencies of the strategies F1 and F2 on the square lattice. The path length per occupied site is denoted by $\alpha(q)$. The lowest line shows the exact slope of $p^{1/2}\alpha(q)$ at $q = 0$.

The strategy F2 works much better in this regime. The corresponding transition matrix $T_2$ is given here explicitly only for $L = 3$

$$T_2 = \begin{pmatrix} q^2 & pq + \frac{1}{2}p^2 & p \\ pq & q^2 & pq \\ p & pq + \frac{1}{2}p^2 & q^2 \end{pmatrix}. \tag{22}$$

The steady state values prob$(r_n)$ are proportional to the right eigenvector with unit eigenvalue and are easily seen to be

$$\text{prob}(r_n = 1) = \text{prob}(r_n = 3) = (1 + q)/(2 + 4q) \tag{23a}$$

$$\text{prob}(r_n = 2) = q/(1 + 2q). \tag{23b}$$

As illustrated in figure 2(*b*), the horizontal distance traversed in the *n*th row depends not only on $x_{n-1}$ and $x_n$ but also on $l_n$ and $r_n$. But once we know the probabilities prob$(r_{n-1})$, we can easily compute the average number of horizontal bonds traversed in the *n*th row for different configurations of sites. We find

$$\alpha_2(q) = (1 - q/2)/(1 - q) \qquad \text{for } L = 2 \tag{24a}$$

$$= (1 + 2pq)/(1 + 2q) \qquad \text{for } L = 3. \tag{24b}$$

Higher $L$ values can be treated similarly. In figure 3 we have plotted the function $\sqrt{p}\alpha_2(q)$ for $p$ between 0.3 and 1. For small $p$, the estimate $\alpha_1(q)$ works almost as well as $\alpha_2(q)$ and in the coninuum limit $p \to 0$, both strategies F1 and F2 are the same; $\sqrt{p}\alpha(q)$ tends to the limit $\sqrt{\frac{4}{3}}$ for both.

## 5. The square lattice for $q$ near zero

For the strategies F1 and F2, the value of the upper bound on $\sqrt{p}\alpha(q)$ is always greater than one. This is not very satisfactory, as the true function $\sqrt{p}\alpha(q)$ is expected to be less than one. (On average, the path is shorter if the cities are randomly distributed than if they are arranged in a regular fashion as in a square grid with lattice spacing $m$ for $p = 1/m^2$, cf (7).) We now show that $\alpha(q)$ has zero slope at $q = 0$, and hence the function $(1-q)^{1/2}\alpha(q)$ is less than one for $q$ sufficiently small.

The strategy we shall use to obtain this upper bound is an improvement on the foliation strategy discussed in the preceding section. The main difference from the earlier strategy is that in this case the paths are not constrained to lie within a strip but are allowed to deform depending on the configuration.

For $q = 0$, the standard route is composed of horizontal strands (or layers) which run through the lattice and which are connected at the ends. For small non-zero $q$, layers are constructed recursively, proceeding from the bottom upwards. Let $L$ denote the last strand to have been constructed. It consists of horizontal segments—which are typically quite long—with occasional vertical steps. The next strand $L'$ consists of all the sites neighbouring those sites in $L$ not yet covered by strands. The construction of segments of $L'$ close to various sorts of segments of $L$ is described by the following rules $(a)$–$(e)$ as depicted in the respective parts $(a)$–$(e)$ of figure 4.

$(a)$ A horizontal stretch of $L$ leads to a horizontal stretch in $L'$, provided no vacant site is encountered.

$(b)$ If there is a vacancy in the row above a horizontal stretch of $L$, the strand $L'$ is deformed so as to avoid the vacancy.

$(c)$ Where there is a vertical step in $L$, the next strand $L'$ is formed by para'iel-shifting $L$ in the manner illustrated.

$(d)$ If there is a depression in $L$, formed by two adjacent vertical steps, the next strand is horizontal.

$(e)$ If there is a depression in $L$, formed by two vertical steps which are two lattice spacings apart, the current strand $L$ is revised in the manner shown and the next strand $L'$ caps the structure.

An application of these rules to a specific configuration of vacant sites is illustrated in figure 5. The resulting TS path differs from the standard route (the path for the case $q = 0$ with all strands horizontal) primarily through the occurrence of vertical steps of height unity. We shall refer to a vertical step which increases (decreases) the height by one as a kink (antikink). It is interesting to identify the rules $(a)$–$(e)$ with the (discrete) dynamics of a set of interacting kinks and antikinks on a line. Thus figure 5 can be viewed as the spacetime diagram describing the evolution of a set of kinks and antikinks (the time direction being along the vertical).

Kink-antikink pairs are created at vacancies (figure 4$(b)$) and since these occur independently with probability $q$ there is a homogeneous (stochastic) rate $q$ of creation in this system. Once generated, a kink moves left one unit deterministically per unit time, and similarly an antikink moves rightwards (figure 4$(c)$). If there is an antikink exactly one step left of a kink, they annihilate each other in the next time step (figure 4$(d)$). If the antikink is two steps left of a kink (figure 4$(e)$) then if they had continued to move according to rule $(c)$ they would be on top of each other at the next time step. This is not allowed, and instead the kink becomes an antikink, and the antikink a kink, and then they move one unit to the right and left respectively according to rule $(c)$.

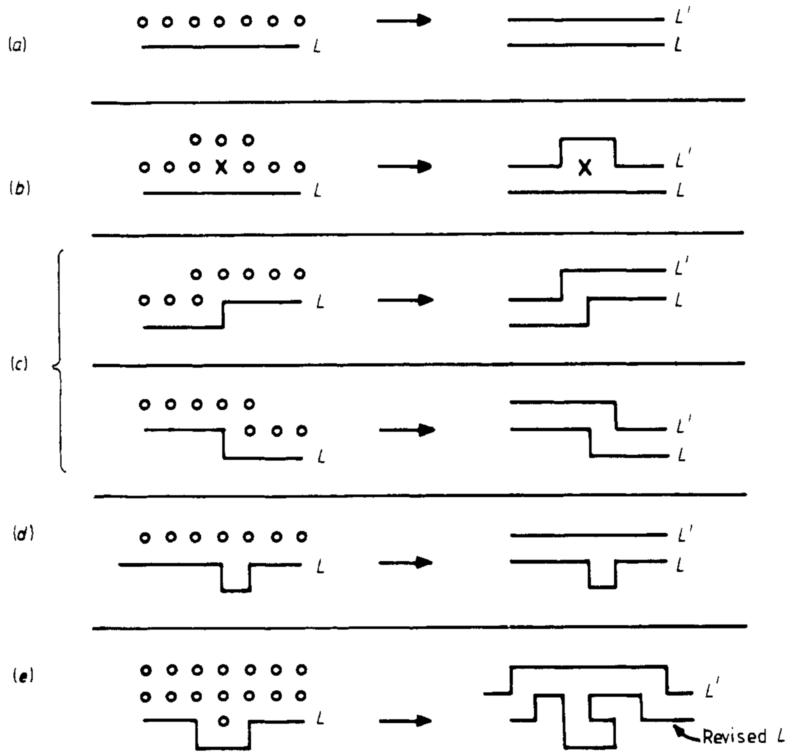


**Figure 4.** Strategy rules for the TSP on the square lattice for $q$ near 0. Cases $(a)$–$(e)$ show the construction of a strand $L'$ in the neighbourhood of different configurations of the previous strand $L$.

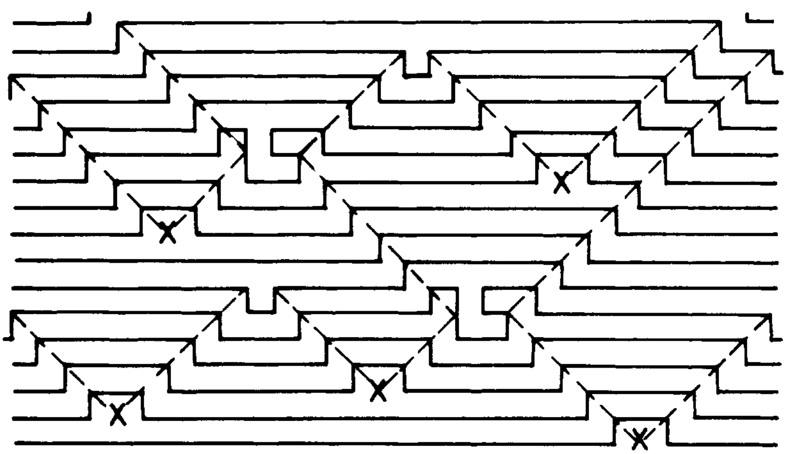

**Figure 5.** The TS strands resulting from the application of the strategy described in figure 4 to a configuration with a small concentration of vacancies (denoted by crosses). The worldlines of kinks and antikinks are shown as broken lines.

It may be noted that in rules $(a)$-$(e)$ we have not dealt with configurations which do not occur to order $q$. These include (i) a cluster of two or more vacancies and (ii) a vacancy close to a kink or antikink. In the application of the strategy, such vacancies are treated as occupied and no attempt is made to avoid them. The number of kinks and antikinks at a given time scales inversely with the intervacancy separation and is proportional to $\sqrt{q}$. Hence the probability of a vacancy occurring in the immediate vicinity of a kink or antikink is $q^{3/2}$. The probability of occurrence of a cluster is even lower (of the order of $q^2$). Isolated vacancies, however, are successfully avoided and we save one step in the path length per vacancy. This gives us equation (13).

The construction of a TS path which would give $\alpha(q)=1+O(q^3)$ on the square lattice remains an open problem. Also a direct proof that $p^{1/d}\alpha(q)\leq 1$ for all $q$ would be very desirable.

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